Proceedings of the Federated Conference on
Computer Science and Information Systems pp. 69–75
DOI: 10.15439/2020F55
ISSN 2300-5963 ACSIS, Vol. 21

Extended Distributive Contact Lattices and Extended Contact Algebras

Tatyana Ivanova
Bulgarian Academy of Sciences
Institute of Mathematics and Informatics
in Sofia
1113, Sofia, Bulgaria, Acad. Georgi Bonchev Str., Block 8
Email: tatyana.ivanova@math.bas.bg

Abstract—**The notion of contact algebra is one of the main tools in mereotopology. This paper considers a generalisation of contact algebra (called extended distributive contact lattice) and the so called extended contact algebras which extend the language of contact algebras by the predicates covering and internal connectedness.**

I. Introduction

IN CLASSICAL Euclidean geometry the notion of point is taken as one of the basic primitive notions. In contrast, region-based theory of space (RBTS) has as primitives the more realistic notion of region (abstraction of physical body) together with some basic relations and operations on regions. Some of these relations are mereological - part–of, overlap and its dual underlap. Other relations are topological - contact, nontangential part-of, dual contact and some others definable by means of the contact and part-of relations. This is one of the reasons that the extension of mereology with these new relations is commonly called mereotopology. There is no clear difference in literature between RBTS and mereotopology. The origin of RBTS goes back to Whitehead and de Laguna ([30], [21]). According to Whitehead points, as well as the other primitive notions in Euclidean geometry such as lines and planes, do not have separate existence in reality and because of this are not appropriate for primitive notions. Survey papers on RBTS are [26], [7], [16], [22] (also the handbook [1] and [5], containing some logics of space).

RBTS has applications in computer science because of its simpler way of representing of qualitative spatial information. Mereotopology is used in the field of Artificial Intelligence, called Knowledge Representation (KR). RBTS initiated a special field in KR, called Qualitative Spatial Representation and Reasoning (QSRR) which is appropriate for automatization [6], [24]. RBTS is applied in geographic information systems, robot navigation. Surveys concerning various applications are for example [8], [9] and the book [17] (also special issues of Fundamenta Informaticae [11] and the Journal of Applied Nonclassical Logics [3]). One of the most popular systems in Qualitative Spatial Rrepresentation and Reasoning is the Region Connection Calculus (RCC) [23].

The notion of contact algebra is one of the main tools in RBTS. This notion appears in the literature under different names and formulations as an extension of Boolean algebra with some mereotopological relations [29], [25], [28], [27], [7], [15], [10], [14]. The simplest system, called just a contact algebra was introduced in [10] as an extension of Boolean algebra $B = (B, 0, 1, \cdot, +, *)$ with a binary relation C called *contact* and satisfying five simple axioms:

(C1) If aCb, then $a \neq 0$,
(C2) If aCb and $a \leq c$ and $b \leq d$, then cCd,
(C3) If $aC(b + c)$, then aCb or aCc,
(C4) If aCb, then bCa,
(C5) If $a \cdot b \neq 0$, then aCb.

The elements of the Boolean algebra are called regions and are considered as analogs of physical bodies. Boolean operations are considered as operations for constructing new regions from given ones. The unit element 1 symbolizes the region containing as its parts all regions, and the zero element 0 symbolizes the empty region.

Topological spaces are among the first mathematical models of space, applied in practice. Standard models of contact algebras are topological. Let X be a topological space and a be its subset. We say that a is regular closed if a is the closure of the interior of a. It is a well known fact that the set $RC(X)$ of all regular closed subsets of X is a Boolean algebra with respect to the following definitions: $a \leq b$ iff $a \subseteq b$, 0 is the empty set, 1 is the set X, $a + b = a \cup b$, $a \cdot b = Cl\,Int\,(a \cap b)$, $a^* = Cl(X \setminus a)$. If we define a contact by taking aCb iff $a \cap b$ is nonempty, then we obtain a contact algebra related to X, namely $\underline{RC(X)} = (RC(X), \leq, 0, 1, \cdot, +, *, C)$ ([10], Example 2.1).

This paper is mostly a summary of the work, contained in [20], [19], [18], [4]. The results, concerning quantifier-free first-order logics for extended contact algebras, are novel and will be submitted as a paper with title "Quantifier-free first-order logics for extended contact algebras".

II. Extended distributive contact lattices (EDCL)

Sometimes there is a problem in the motivation of the operation Boolean complement ($*$) of contact algebra. A question arises - if a represents some region, what region does a^* represent - it depends on the universe in which we consider a. Moreover if a represents a physical body, then a^* is unnatural - such a physical body does not exist. Because of this we can drop the operation of complement and replace the Boolean part of a contact algebra with distributive lattice. First steps in this direction were made in [12], [13], introducing the notion of distributive contact lattice. In a distributive contact lattice the only mereotopological relation is the contact relation. Non-tangential inclusion and dual contact are not included in the language. In [20], the language of distributive contact lattices is extended by considering these two relations as nondefinable primitives. An axiomatization is obtained of the theory consisting of the universal formulas in this more expressive language, true in all contact algebras. The structures, satisfying the axioms in question, are called extended distributive contact lattices (EDCL). The well known RCC-8 system of mereotopological relations is definable in the language of EDCL and is not definable in the language of distributive contact lattices.

EDCL is a generalization of contact algebra, defined in the following way:

Definition 2.1: [20] **Extended distributive contact lattice.** Let $\underline{D} = (D, \leq, 0, 1, \cdot, +, C, \widehat{C}, \ll)$ be a bounded distributive lattice with three additional relations $C, \widehat{C}, \ll$, called respectively *contact*, *dual contact* and *nontangential part-of*. The obtained system, denoted shortly by $\underline{D} = (D, C, \widehat{C}, \ll)$, is called *extended distributive contact lattice* (EDCL, for short) if it satisfies the axioms listed below.

Notations: if R is one of the relations $\leq, C, \widehat{C}, \ll$, then its complement is denoted by $\overline{R}$.

Axioms for C alone: The axioms (C1)-(C5) mentioned above.

Axioms for $\widehat{C}$ alone:

($\widehat{C}$1) If $a\widehat{C}b$, then $a, b \neq 1$,
($\widehat{C}$2) If $a\widehat{C}b$ and $a' \leq a$ and $b' \leq b$, then $a'\widehat{C}b'$,
($\widehat{C}$3) If $a\widehat{C}(b \cdot c)$, then $a\widehat{C}b$ or $a\widehat{C}c$,
($\widehat{C}$4) If $a\widehat{C}b$, then $b\widehat{C}a$,
($\widehat{C}$5) If $a + b \neq 1$, then $a\widehat{C}b$.

Axioms for $\ll$ alone:

($\ll$ 1) $0 \ll 0$,
($\ll$ 2) $1 \ll 1$,
($\ll$ 3) If $a \ll b$, then $a \leq b$,
($\ll$ 4) If $a' \leq a \ll b \leq b'$, then $a' \ll b'$,
($\ll$ 5) If $a \ll c$ and $b \ll c$, then $(a + b) \ll c$,
($\ll$ 6) If $c \ll a$ and $c \ll b$, then $c \ll (a \cdot b)$,
($\ll$ 7) If $a \ll b$ and $(b \cdot c) \ll d$ and $c \ll (a + d)$, then $c \ll d$.

Mixed axioms:

$(MC1)$ If aCb and $a \ll c$, then $aC(b \cdot c)$,
$(MC2)$ If $a\overline{C}(b \cdot c)$ and aCb and $(a \cdot d)\overline{C}b$, then $d\widehat{C}c$,
$(M\widehat{C}1)$ If $a\widehat{C}b$ and $c \ll a$, then $a\widehat{C}(b + c)$,
$(M\widehat{C}2)$ If $a\overline{\widehat{C}}(b + c)$ and $a\widehat{C}b$ and $(a + d)\overline{\widehat{C}}b$, then dCc,
$(M \ll 1)$ If $a\overline{\widehat{C}}b$ and $(a \cdot c) \ll b$, then $c \ll b$,
$(M \ll 2)$ If $a\overline{C}b$ and $b \ll (a + c)$, then $b \ll c$.

Lemma 2.2: [20] Let (W, R) be a relational system with reflexive and symmetric relation R and let $\underline{D}$ be any collection of subsets of W which is a bounded distributive set-lattice with relations $C, \widehat{C}$ and $\ll$ defined as follows:
(Def C_R) aC_Rb iff $\exists x \in a$ and $\exists y \in b$ such that xRy;
(Def $\widehat{C}_R$) $a\widehat{C}_Rb$ iff $\exists x \notin a$ and $\exists y \notin b$ such that xRy;
(Def $\not\ll_R$) $a \not\ll_R b$ iff $\exists x \in a$ and $\exists y \notin b$ such that xRy.
Then $(\underline{D}, C_R, \widehat{C}_R, \ll_R)$ is an EDCL.

EDCL $\underline{D} = (D, C_R, \widehat{C}_R, \ll_R)$ over a relational system (W, R) is called *discrete EDCL*. If D is a set of all subsets of W then $\underline{D}$ is called a *full discrete EDCL*.

Corollary 2.3: [20] The axioms of the relations $C, \widehat{C}$ and $\ll$ are true in contact algebras.

Generalizing the Stone representation theorem for distributive lattices it is proved the following theorem.

Theorem 2.4: [20] **Relational representation theorem of EDCL.** Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDCL. Then there is a relational system $\underline{W} = (W, R)$ with reflexive and symmetric R and an embedding h into the EDCL of all subsets of W.

Corollary 2.5: [20] Every EDCL can be isomorphically embedded into a contact algebra.

In [20], it is obtained a new stronger form of the well-known in the theory of distributive lattices Filter-extension lemma. This stronger form is equivalent to the Axiom of Choice. This stronger form is used in the proof of the relational representation theorem for EDCL.

Lemma 2.6: [20] **Strong filter-extension Lemma.** Let F_0 be a filter, I_0 be an ideal and $F_0 \cap I_0 = \emptyset$. Then there exists a prime filter F such that $F_0 \subseteq F$, $(\forall x \in F)(x \notin I_0)$ and $(\forall x \notin F)(\exists y \in F)(x \cdot y \in I_0)$.

III. Topological representation theory of EDCL

In [20], are considered also some axiomatic extensions of EDCL yielding representations in T_1 and T_2 topological spaces.

Several additional axioms for EDCL are formulated which are adaptations for the language of EDCL of some known axioms considered in the context of contact algebras. The first new axioms for EDCL are the so called extensionality axioms for the definable predicates of overlap - $aOb \leftrightarrow_{def} a \cdot b \neq 0$ and underlap - $a\widehat{O}b \leftrightarrow_{def} a + b \neq 1$.
(Ext O) $a \not\leq b \rightarrow (\exists c)(a \cdot c \neq 0$ and $b \cdot c = 0)$ - *extensionality of overlap*,
(Ext $\widehat{O}$) $a \not\leq b \rightarrow (\exists c)(a + c = 1$ and $b + c \neq 1)$ - *extensionality of underlap*.

We say that a lattice is *O-extensional* if it satisfies (Ext O) and *U-extensional* if it satisfies (Ext $\widehat{O}$). Note that the conditions (Ext O) and (Ext $\widehat{O}$) are true in Boolean algebras but not always are true in distributive lattices.

The following additional axioms are considered too:

(Ext C) $a \neq 1 \rightarrow (\exists b \neq 0)(a\overline{C}b)$ - *C-extensionality*,

(Ext $\widehat{C}$) $a \neq 0 \rightarrow (\exists b \neq 1)(a\overline{\widehat{C}}b)$ - *$\widehat{C}$-extensionality*,

(Con C) $a \neq 0, b \neq 0$ and $a+b=1 \rightarrow aCb$ - *C-connectedness axiom* ,

(Con $\widehat{C}$) $a \neq 1, b \neq 1$ and $a \cdot b = 0 \rightarrow a\widehat{C}b$ - *$\widehat{C}$-connectedness axiom* ,

(Nor 1) $a\overline{C}b \rightarrow (\exists c, d)(c+d=1, a\overline{C}c$ and $b\overline{C}d)$,

(Nor 2) $a\overline{\widehat{C}}b \rightarrow (\exists c, d)(c \cdot b = 0, a\overline{\widehat{C}}c$ and $b\overline{\widehat{C}}d)$,

(Nor 3) $a \ll b \rightarrow (\exists c)(a \ll c \ll b)$.

(U-rich $\ll$) $a \ll b \rightarrow (\exists c)(b+c=1$ and $a\overline{C}c)$,

(U-rich $\widehat{C}$) $a\overline{\widehat{C}}b \rightarrow (\exists c, d)(a+c=1, b+d=1$ and $c\overline{C}d)$,

(O-rich $\ll$) $a \ll b \rightarrow (\exists c)(a \cdot c = 0$ and $c\overline{\widehat{C}}b)$,

(O-rich C) $a\overline{C}b \rightarrow (\exists c, d)(a \cdot c = 0, b \cdot d = 0$ and $c\overline{\widehat{C}}d)$.

Let $(D_1, C_1, \widehat{C}_1, \ll_1)$ and $(D_2, C_2, \widehat{C}_2, \ll_2)$ be two EDCL and D_1 is a substructure of D_2. It is valuable to know under what conditions we have equivalences of the form:

D_1 satisfies some additional axiom iff D_2 satisfies the same axiom.

Remark 3.1: [20] The importance of such conditions is related to the representation theory of EDCL satisfying some additional axioms. In general, if we have some embedding theorem for EDCL D satisfying a given additional axiom A, it is not known in advance that the lattice in which D is embedded also satisfies A. That is why it is good to have such conditions which automatically guarantee this. Below several such "good conditions" are formulated: dense and dual dense sublattice, C-separable sublattice.

Definition 3.1: [20] **Dense and dual dense sublattice.** Let D_1 be a distributive sublattice of D_2. D_1 is called a *dense* sublattice of D_2 if the following condition is satisfied:

(Dense) $(\forall a_2 \in D_2)(a_2 \neq 0 \Rightarrow (\exists a_1 \in D_1)(a_1 \leq a_2$ and $a_1 \neq 0))$.

If h is an embedding of the lattice D_1 into the lattice D_2 then we say that h is a *dense* embedding if the sublattice $h(D_1)$ is a dense sublattice of D_2.

Dually, D_1 is called a *dual dense* sublattice of D_2 if the following condition is satisfied:

(Dual dense) $(\forall a_2 \in D_2)(a_2 \neq 1 \Rightarrow (\exists a_1 \in D_1)(a_2 \leq a_1$ and $a_1 \neq 1))$.

If h is an embedding of the lattice D_1 into the lattice D_2 then we say that h is a *dual dense* embedding if the sublattice $h(D_1)$ is a dually dense sublattice of D_2.

(See [13] for some known characterizations of density and dual density in distributive lattices.)

For the case of contact algebras [26] and distributive contact lattices [13] the notion of C-separability is introduced as follows. Let D_1 be a substructure of D_2; we say that D_1 is a C-separable sublattice of D_2 if the following condition is satisfied:

(C-separable) $(\forall a_2, b_2 \in D_2)(a_2\overline{C}b_2 \Rightarrow (\exists a_1, b_1 \in D_1)(a_2 \leq a_1, b_2 \leq b_1, a_1\overline{C}b_1))$.

For the case of EDCL this notion is modified, adding two additional clauses corresponding to the relations $\widehat{C}$ and $\ll$ just having in mind the definitions of these relations in contact algebras. Namely

Definition 3.2: [20] **C-separability.** Let D_1 be a substructure of D_2; we say that D_1 is a *C-separable EDC-sublattice of* D_2 if the following conditions are satisfied:

(C-separability for C) -
$(\forall a_2, b_2 \in D_2)(a_2\overline{C}b_2 \Rightarrow (\exists a_1, b_1 \in D_1)(a_2 \leq a_1, b_2 \leq b_1, a_1\overline{C}b_1))$.

(C-separability for $\widehat{C}$) -
$(\forall a_2, b_2 \in D_2)(a_2\overline{\widehat{C}}b_2 \Rightarrow (\exists a_1, b_1 \in D_1)(a_2 + a_1 = 1, b_2 + b_1 = 1, a_1\overline{C}b_1))$.

(C-separability for $\ll$) -
$(\forall a_2, b_2 \in D_2)(a_2 \ll b_2 \Rightarrow (\exists a_1, b_1 \in D_1)(a_2 \leq a_1, b_2 + b_1 = 1, a_1\overline{C}b_1))$.

If h is an embedding of the lattice D_1 into the lattice D_2 then we say that h is a *C-separable embedding* if the sublattice $h(D_1)$ is a C-separable sublattice of D_2.

Theorem 3.3: [20] **Topological representation theorem for EDCL.** Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDCL. Then there exists a topological space X and an embedding of $\underline{D}$ into the contact algebra $RC(X)$ of regular closed subsets of X.

Definition 3.4: [20] **U-rich and O-rich EDCL.** Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDCL. Then:

(i) $\underline{D}$ is called U-rich EDCL if it satisfies the axioms (Ext $\widehat{O}$), (U-rich $\ll$) and (U-rich $\widehat{C}$).

(ii) $\underline{D}$ is called O-rich EDCL if it satisfies the axioms (Ext O), (O-rich $\ll$) and (O-rich $\widehat{C}$).

In [20], is developed the topological representation theory of U-rich EDCL. In a dual way can be developed the topological representation theory of O-rich EDCL.

Theorem 3.5: [20] **Topological representation theorem for U-rich EDCL.**
Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an U-rich EDCL. Then there exists a compact semiregular T_0-space X and a dually dense and C-separable embedding h of $\underline{D}$ into the Boolean contact algebra $RC(X)$ of the regular closed sets of X. Moreover:

(i) $\underline{D}$ satisfies (Ext C) iff $RC(X)$ satisfies (Ext C); in this case X is weakly regular.

(ii) $\underline{D}$ satisfies (Con C) iff $RC(X)$ satisfies (Con C); in this case X is connected.

(iii) $\underline{D}$ satisfies (Nor 1) iff $RC(X)$ satisfies (Nor 1); in this case X is κ-normal.

There is also a topological representation theorem of U-rich EDCL, satisfying (Ext C), in T_1-spaces.

Adding the axiom (Nor 1), it is obtained representability in compact T_2-spaces.

IV. Logics for EDCL

In [19], are considered a logic for EDCL and several extending it logics, corresponding to topological spaces possessing various additional properties. Completeness theorems are given with respect to both algebraic and topological semantics for these logics. It turns out that they are decidable.

It is considered the quantifier-free first-order language $\mathcal{L}$ which includes:
• constants: 0, 1;
• function symbols: $+$, $\cdot$;
• predicate symbols: $\leq$, C, $\widehat{C}$, $\ll$.
Every EDCL is a structure for $\mathcal{L}$.

It is considered the logic L with rule MP and the following axioms:
• the axioms of the classical propositional logic;
• the axiom schemes of distributive lattice;
• the axioms for C, $\widehat{C}$, $\ll$ and the mixed axioms of EDCL - considered as axiom schemes.

The following additional rules and an axiom scheme are considered:

(R Ext $\widehat{O}$) $\frac{\alpha \to (a+p \neq 1 \vee b+p=1) \text{ for all variables } p}{\alpha \to (a \leq b)}$, where α is a formula, a, b are terms

(R U-rich $\ll$) $\frac{\alpha \to (b+p \neq 1 \vee aCp) \text{ for all variables } p}{\alpha \to (a \ll b)}$, where α is a formula, a, b are terms

(R U-rich $\widehat{C}$) $\frac{\alpha \to (a+p \neq 1 \vee b+q \neq 1 \vee pCq) \text{ for all variables } p,\ q}{\alpha \to a\widehat{C}b}$, where α is a formula, a, b are terms

(R Ext C) $\frac{\alpha \to (p \neq 0 \to aCp) \text{ for all variables } p}{\alpha \to (a=1)}$, where α is a formula, a is a term

(R Nor1) $\frac{\alpha \to (p+q \neq 1 \vee aCp \vee bCq) \text{ for all variables } p,\ q}{\alpha \to aCb}$, where α is a formula, a, b are terms

(Con C) $p \neq 0 \wedge q \neq 0 \wedge p+q=1 \to pCq$

The additional axioms for EDCL (the axioms (Ext $\widehat{O}$), (U-rich $\ll$), (U-rich $\widehat{C}$), (Ext C), (Nor 1)) correspond to these rules.

Let L' be for example the extension of L with the rule (R Ext $\widehat{O}$) and the axiom scheme (Con C). Then we denote L' by $L_{ConC,Ext\widehat{O}}$ and call the axioms (Con C) and (Ext $\widehat{O}$) additional axioms, corresponding to L'. In a similar way we denote any extension of L with some of the considered additional rules and axiom scheme and in a similar way we define its corresponding additional axioms.

The following theorem is true

Theorem 4.1: [19] **Completeness theorem with respect to algebraic semantics.** Let L' be some extension of L with zero or more of the considered additional rules and axiom scheme. The following conditions are equivalent for any formula α:
(i) α is a theorem of L';
(ii) α is true in all EDCL, satisfying the additional axioms, corresponding to L'.

To every of the logics
1) L;
2) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$;
3) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC}$;
4) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC}$;
5) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1}$;
6) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC}$;
7) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1,ConC}$;
8) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,Nor1}$;
9) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC,Nor1}$.

is juxtaposed a class of topological spaces:
1) the class of all T_0, semiregular, compact topological spaces;
2) the class of all T_0, semiregular, compact topological spaces;
3) the class of all T_0, compact, weakly regular topological spaces;
4) the class of all T_0, semiregular, compact, connected topological spaces;
5) the class of all T_0, semiregular, compact, κ - normal topological spaces;
6) the class of all T_0, compact, weakly regular, connected topological spaces;
7) the class of all T_0, semiregular, compact, κ - normal, connected topological spaces;
8) the class of all T_0, compact, weakly regular, κ - normal topological spaces;
9) the class of all T_0, compact, weakly regular, connected, κ - normal topological spaces.

We have the following theorems

Theorem 4.2: [19] **Completeness theorem with respect to topological semantics.** Let L' be any of the considered above logics. The following conditions are equivalent for any formula α:
(i) α is a theorem of L';
(ii) α is true in all contact algebras over a topological space from the class, corresponding to L'.

Theorem 4.3: [19] (i) The logics
L,
$L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$,
$L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC}$,
$L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1}$,
$L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,Nor1}$
have the same theorems and are decidable;
(ii) The logics
$L_{ConC,U-rich\ll}$,
$L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC}$,
$L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC,Nor1}$,
$L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC}$,
$L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC,Nor1}$
have the same theorems and are decidable.

V. Extended contact algebras

The predicate internal connectedness (intuitively meaning that the interior is connected) cannot be defined in the language of contact algebras ([18], Proposition 2.1). So we consider *extended contact algebras:*

Definition 5.1: [18] **Extended contact algebra (ExtCA, for short)** is a system $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$, where $(B, \leq, 0, 1, \cdot, +, *)$ is a nondegenerate Boolean algebra, $\vdash$ (*covering* or *extended contact*) is a ternary relation in B such that the following axioms are true:
(1) $a, b \vdash c \to b, a \vdash c$,
(2) $a \leq c \to a, b \vdash c$,
(3) $a, b \vdash x,\ a, b \vdash y,\ x, y \vdash c \to a, b \vdash c$,
(4) $a, b \vdash c \to a \cdot b \leq c$,
(5) $a, b \vdash c \to a + x, b \vdash c + x$,
C is a binary relation in B such that
(6) $aCb \leftrightarrow a, b \not\vdash 0$,
c^o (internal connectedness) is a unary predicate in B such that
(7) $c^o(a) \leftrightarrow \forall b \forall c(b \neq 0 \wedge c \neq 0 \wedge a = b + c \to b, c \not\vdash a^*)$.

ExtCAs extend the language of contact algebras by the predicate covering and the predicate internal connectedness. The internal connectedness is defined by the relation of covering ($c^o(a)$ iff $\forall b \forall c(b \neq 0 \wedge c \neq 0 \wedge a = b + c \to b, c \not\vdash a^*)$ ([18], Proposition 3.1)). Another motivation for considering the relation of covering is that by it we can define the property of two regions their intersection to be a region. Extended contact gives also the possibility to define the relation of contact. One of the motivations for adding the predicate internal connectedness is that by its help the property "existing of cavities in a physical body" can be defined: we have "a has cavities" if and only if "a^* is not internally connected". We cannot define "a has cavities" if and only if "the complement of a is not connected", using the predicate connectedness because the complement of a is not necessarily regular closed set i.e. element of the topological model of ExtCA. If we define "a has cavities" if and only if "a^* is not connected", this is wrong - if the cavity in the ball a touches its boundary, a^* is connected (and at the same time is not internally connected). Because of these reasons we need the predicate "internal connectedness" instead of "connectedness" for defining the property "existing of cavities in a physical body".

Primary semantics for ExtCAs is topological. Let X be a topological space. A topological ExtCA over X is the structure with universe the set $RC(X)$ of all regular closed subsets together with the following interpretations: $a \leq b$ iff $a \subseteq b$, $0 = \emptyset$, $1 = X$, $a \cdot b = Cl\, Int\,(a \cap b)$, $a + b = a \cup b$, $a^* = Cl\,(X \setminus a)$, $a, b \vdash c$ iff $a \cap b \subseteq c$, aCb iff $a, b \not\vdash \emptyset$, $c^o(a)$ iff $Int\, a$ is a connected subspace of X.

We have the following

Theorem 5.2: [18] **Topological representation theorem.** Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ be an ExtCA. Then there is a compact, semiregular, T_0 topological space X and an embedding of $\underline{B}$ into the topological ExtCA over X.

It is interesting also to consider a relational semantics for ExtCAs. This is done in [4].

Definition 5.3: [4] An *equivalence frame of type 2* is a relational structure of the form (W, R_1, R_2), where W is a nonempty set and R_1 and R_2 are equivalence relations on W.

Definition 5.4: [4] Let (W, R_1, R_2) be an equivalence frame of type 2. A *relational ExtCA over* (W, R_1, R_2) is the structure: $\underline{B} = (2^W, \subseteq, \emptyset, W, \cap, \cup, *, \vdash, C, c^o)$, where $*$ denotes the set theoretical complement and for any subsets of W a, b, and c:

- $a, b \vdash c$ iff $\forall A, A_1, B, B_1\Big(AR_1A_1 \in a, BR_1B_1 \in b, AR_2B \to (\exists C, C_1)(CR_1C_1 \in c, AR_2C)\Big)$ and $a \cap b \subseteq c$,
- aCb iff $a, b \not\vdash \emptyset$,
- $c^o(a)$ iff $(\forall b, c \subseteq W)(b \neq \emptyset, c \neq \emptyset, a = b \cup c \to b, c \not\vdash (W \setminus a))$.

We say that a formula is true in (W, R_1, R_2) if it is true in the ExtCA over (W, R_1, R_2).

It turns out that the internal connectedness in a relational ExtCA means the following (see Figure 1):
$c^o(a)$ if and only if $(\forall b, c \subseteq W)(b, c \neq \emptyset$ and $a = b \cup c \to b \cap c \neq \emptyset$ or
$(\exists A, A_1, B, B_1)(AR_1A_1 \in b,\ BR_1B_1 \in c,\ AR_2B,$ $(\forall C, C_1)(AR_2C, BR_2C, CR_1C_1 \to C_1 \in a)))$

We have the following

Theorem 5.5: [4] **Relational representation theorem.** Let $\underline{B}$ be a finite ExtCA. Then $\underline{B}$ is isomorphically embedded in the relational ExtCA over some equivalence frame of type 2 (W, R_1, R_2).

We consider a quantifier-free first-order logic $\mathbb{L}$ for ExtCAs which has the following:

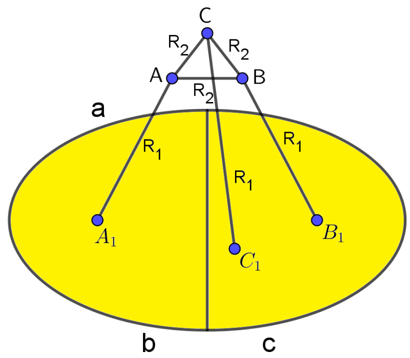

Fig. 1. **Internal connectedness in a relational ExtCA**

• axioms:
- the axioms of the classical propositional logic;
- the axioms of Boolean algebra;
- the axioms of ExtCA concerning the relations extended contact and contact;
- the axiom schemes:
(Ax c^o) $c^o(p) \wedge q \neq 0 \wedge r \neq 0 \wedge p = q + r \rightarrow q, r \nvdash p^*$
(Ax c^o 1) $c^o(0)$
(Ax c^o 2) $\neg c^o(p+q) \rightarrow \neg c^o(p) \vee \neg c^o(q)$
(Ax c^o 3) $c^o(p+q) \rightarrow c^o(p) \wedge c^o(q)$
• rules:
- MP

This logic is decidable and we have the following

Theorem 5.6: **Completeness theorem with respect to relational semantics.** For every quantifier-free formula α the following conditions are equivalent:
i) α is a theorem of $\mathbb{L}$;
ii) α is true in all equivalence frames of type 2.

Theorem 5.7: **Completeness theorem with respect to topological and algebraic semantics.** For every quantifier-free formula α the following conditions are equivalent:
i) α is a theorem of $\mathbb{L}$;
ii) α is true in all ExtCAs;
iii) α is true in all topological ExtCAs over a compact, T_0, semiregular topological space.

Extended contact gives also the possibility to define the relation of contact (aCb iff $a, b \nvdash 0$) and the binary relation $RC_\cap$ meaning that the intersection of two regular closed sets is a regular closed set ($RC_\cap(a, b)$ iff $a, b \vdash a \cdot b$). It is worth to consider also a quantifier-free first-order language without the predicate of internal connectedness i.e. $\mathcal{L}(0, 1; \cdot, +, *; \leq, \vdash, C)$. In this weaker language one equivalence relation is enough - we consider *equivalence frames of type 1*:

Definition 5.8: [4] An *equivalence frame of type 1* is a relational structure of the form (W, R), where W is a nonempty set and R is an equivalence relation on W.

Definition 5.9: [4] Let (W, R) be an equivalence frame of type 1. A *relational ExtCA over* (W, R) *in* $\mathcal{L}$ is the structure $\underline{B} = (2^W, \subseteq, \emptyset, W, \cap, \cup, *, \vdash, C)$, where $*$ denotes the set theoretical complement and for any subsets of W a, b, and c:
- $a, b \vdash c$ iff $\Big((\exists A \in a)(\exists B \in b)ARB \rightarrow (\exists C \in c)ARC\Big)$ and $a \cap b \subseteq c$,
- aCb iff $a, b \nvdash \emptyset$

Theorem 5.10: [4] **Relational representation theorem.** Let $\underline{B}$ be a finite ExtCA. Then in $\mathcal{L}$ $\underline{B}$ is isomorphically embedded in the relational ExtCA over some equivalence frame of type 1 (W, R).

This representation theorem is only for finite ExtCA. Trying to overcome this drawback, we define:

Definition 5.11: [4] A *weak extended contact algebra* is a structure of the form $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash)$, where $(B, \leq, 0, 1, \cdot, +, *)$ is a non-degenerate Boolean algebra and $\vdash$ is a ternary relation on B such that for all $a, b, d, e, f \in B$,
(1) if $a \leq d$, $b \leq e$ and $d, e \vdash f$, then $a, b \vdash f$,
(2) if $a = 0$ or $b = 0$, then $a, b \vdash f$,
(3) if $a, b \vdash f$ and $d, e \vdash f$, then $a \cdot d, b + e \vdash f$ and $a + d, b \cdot e \vdash f$,
(4) if $a, b \vdash d$ and $d \leq f$, then $a, b \vdash f$.

Obviously, every extended contact algebra is also a weak extended contact algebra. The converse is not true.

Definition 5.12: [4] A *parametrized frame* is a structure of the form (W, R), where W is a nonempty set and R is a function associating to each subset of W a binary relation on W.

Definition 5.13: [4] Let (W, R) be a parametrized frame. A *relational weak ExtCA over* (W, R) is the structure $\underline{B} = (2^W, \subseteq, \emptyset, W, \cap, \cup, *, \vdash)$, where $*$ denotes the set theoretical complement and $\vdash$ is the ternary relation on W's powerset defined by
- $a, b \vdash d$ iff for all $S \in a$, $T \in b$ and $u \subseteq W$, if $d \subseteq u$, then $(S, T) \notin R(u)$.

Theorem 5.14: [4] **Relational representation theorem.** Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash)$ be a weak ExtCA. Then $\underline{B}$ is isomorphically embedded in the relational weak ExtCA over some parametrized frame (W, R).

Thus we obtain in $\mathcal{L}$ a relational representation theorem for all ExtCA, not only finite (because every ExtCA is a weak ExtCA), but the structure in which we embed is not an

ExtCA and the parametrized frame it is based on is a relatively complex relational structure.

Let $\mathbb{L}_1$ be the logic obtained from $\mathbb{L}$ by removing axioms (Ax c^o), (Ax c^o 1), (Ax c^o 2) and (Ax c^o 3). This logic is called *extended contact logic*. It is decidable and we have the following

Theorem 5.15: **Completeness theorem with respect to relational semantics.** For every formula α in $\mathcal{L}$ the following conditions are equivalent:
i) α is a theorem of $\mathbb{L}_1$;
ii) α is true in all equivalence frames of type 1.

VI. Conclusion

Possible future research directions are for example:

• the complexity of the considered logics;

• to be obtained representation theorems in Euclidean spaces;

• generalization of Theorems 5.5 and 5.10 for all ExtCAs, not only for finite;

• to be obtained a stronger form of Theorem 5.14, where we embed in an ExtCA and the relational structure is simpler.

• in reference to temporal reasoning, if we add to the language of EDCL the binary relation $P(X, Y)$, meaning that the start of time interval X is before the start of time interval Y, then we obtain a language rich enough to define all possible relations between two intervals of Allen's interval algebra ([2]).

Acknowledgements. I thank the anonymous referees for their valuable remarks which improved the readability of the paper. This paper is supported by contract DN02/15/19.12.2016 "Space, Time and Modality: Relational, Algebraic and Topological Models" with Bulgarian NSF.

References

[1] M. Aiello, I. Pratt-Hartmann and J. van Benthem (Eds.), *Handbook of spatial logics.* Springer, 2007.

[2] J. F. Allen, "Maintaining knowledge about temporal intervals," *Communications of the ACM,* vol. 26, (11), 1983, pp. 832–843.

[3] P. Balbiani (Ed.), *Special Issue on Spatial Reasoning, J. Appl. Non-Classical Logics,* vol. 12, (3-4), 2002.

[4] P. Balbiani and T. Ivanova, "Relational representation theorems for extended contact algebras," *Stud Logica,* to appear, also available online with different title: https://arxiv.org/abs/1901.10367

[5] P. Balbiani, T. Tinchev and D. Vakarelov, "Modal logics for region-based theory of space," *Fundamenta Informaticae, Special Issue: Topics in Logic, Philosophy and Foundation of Mathematics and Computer Science in Recognition of Professor Andrzej Grzegorczyk,* vol. 81, (1-3), 2007, pp. 29–82.

[6] B. Bennett, "Determining consistency of topological relations," *Constraints,* vol. 3, 1998, pp. 213–225.

[7] B. Bennett and I. Düntsch, "Axioms, algebras and topology," *in Handbook of Spatial Logics,* M. Aiello, I. Pratt, and J. van Benthem (Eds.), Springer, 2007, pp. 99–160.

[8] A. Cohn and S. Hazarika, "Qualitative spatial representation and reasoning: An overview," *Fuandamenta informaticae*, vol. 46, 2001, pp. 1–20.

[9] A. Cohn and J. Renz, "Qualitative spatial representation and reasoning," *in F. van Hermelen, V. Lifschitz and B. Porter (Eds.) Handbook of Knowledge Representation,* Elsevier, 2008, pp. 551–596.

[10] G. Dimov and D. Vakarelov, "Contact algebras and region–based theory of space: A proximity approach I," *Fundamenta Informaticae,* vol. 74, (2-3), 2006, pp. 209–249.

[11] I. Düntsch (Ed.), *Special issue on Qualitative Spatial Reasoning, Fundam. Inform.,* vol. 46, 2001.

[12] I. Düntsch, W. MacCaull, D. Vakarelov and M. Winter, "Topological representation of contact lattices," *Lecture Notes in Computer Science,* vol. 4136, 2006, pp. 135–147.

[13] I. Düntsch, W. MacCaull, D. Vakarelov and M. Winter, "Distributive contact lattices: Topological representation," *Journal of logic and Algebraic Programming*, vol. 76, 2008, pp. 18–34.

[14] I. Düntsch and D. Vakarelov, "Region-based theory of discrete spaces: A proximity approach," *in M. Nadif, A. Napoli, E. SanJuan and A. Sigayret (Eds.) Proceedings of Fourth International Conference Journées de l'informatique Messine,* Metz, France, 2003, pp. 123–129, *Journal version in Annals of Mathematics and Artificial Intelligence,* vol. 49, (1-4), 2007, pp. 5–14.

[15] I. Düntsch and M. Winter, "A representation theorem for Boolean contact algebras," *Theoretical Computer Science (B),* vol. 347, 2005, pp. 498–512.

[16] T. Hahmann and M. Gruninger, "Region-based theories of space: Mereotopology and beyond," *S. Hazarika (ed.): Qualitative Spatio-Temporal Representation and Reasoning: Trends and Future Directions,* 2012, pp. 1–62, IGI Publishing.

[17] *Qualitative spatio-temporal representation and reasoning: Trends and future directions.* S. M. Hazarika (Ed.), IGI Global, 1st ed., 2012.

[18] T. Ivanova, "Extended contact algebras and internal connectedness," *Stud Logica,* vol. 108, 2020, pp. 239–254.

[19] T. Ivanova, "Logics for extended distributive contact lattices," *Journal of Applied Non-Classical Logics,* vol. 28(1), 2018, pp. 140–162.

[20] T. Ivanova and D. Vakarelov, "Distributive mereotopology: extended distributive contact lattices," *Annals of Mathematics and Artificial Intelligence,* vol. 77(1), 2016, pp. 3–41.

[21] T. de Laguna, "Point, line and surface as sets of solids," *J. Philos,* vol. 19, 1922, pp. 449–461.

[22] I. Pratt-Hartmann, "First-order region-based theories of space," *in Logic of Space, M. Aiello, I. Pratt-Hartmann and J. van Benthem (Eds.),* Springer, 2007.

[23] D. A. Randell, Z. Cui, and A. G. Cohn., "A spatial logic based on regions and connection," *in B. Nebel, W. Swartout, C. Rich (Eds.) Proceedings of the 3rd International Conference Knowledge Representation and Reasoning*, Morgan Kaufmann, Los Allos, CA, 1992, pp. 165–176.

[24] J. Renz and B. Nebel, "On the complexity of qualitative spatial reasoning: a maximal tractable fragment of the region connection calculus," *Artificial Intelligence,* vol. 108, 1999, pp. 69–123.

[25] J. Stell, "Boolean connection algebras: A new approach to the Region Connection Calculus," *Artif. Intell.,* vol. 122, 2000, pp. 111–136.

[26] D. Vakarelov, "Region-based theory of space: Algebras of regions, representation theory and logics," *in D. Gabbay, S. Goncharov and M. Zakharyaschev (Eds.) Mathematical Problems from Applied Logic II. Logics for the XXIst Century,* Springer, 2007, pp. 267–348.

[27] D. Vakarelov, G. Dimov, I. Düntsch, and B. Bennett, "A proximity approach to some region based theory of space," *Journal of applied non-classical logics,* vol. 12, (3-4), 2002, pp. 527–559.

[28] D. Vakarelov, I. Düntsch and B. Bennett, "A note on proximity spaces and connection based mereology," *in C. Welty and B. Smith (Eds.) Proceedings of the 2nd International Conference on Formal Ontology in Information Systems (FOIS'01)*, ACM, 2001, pp. 139–150.

[29] H. de Vries, "Compact spaces and compactifications," Van Gorcum, 1962.

[30] A. N. Whitehead, "Process and Reality," New York, MacMillan, 1929.